Econ 802
Final Exam
Answer Key

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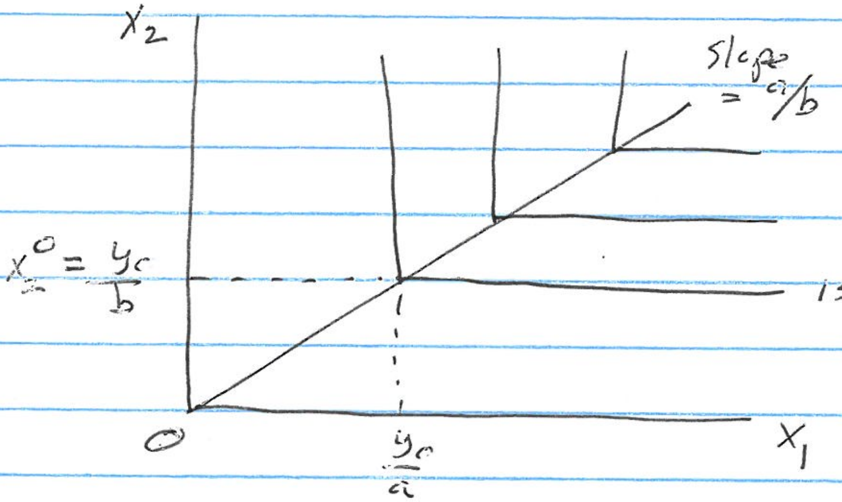
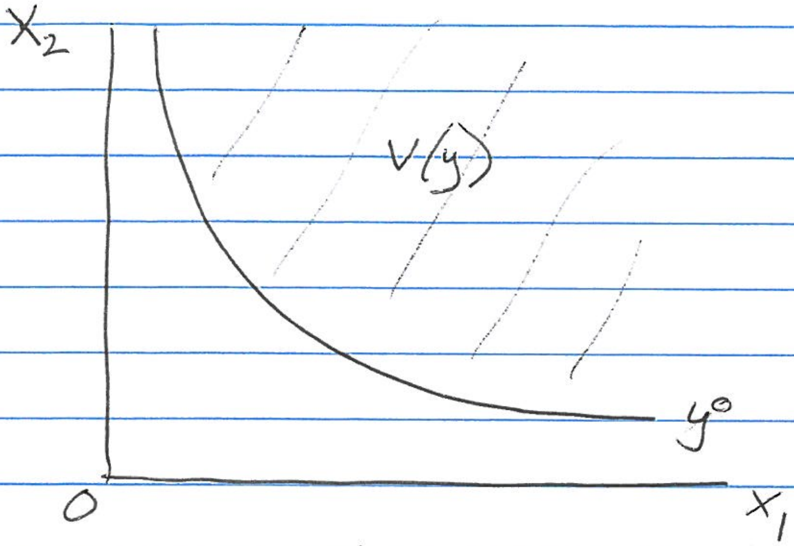
1. (a) The function is $y = \min\{ax_1, bx_2\}$ with $a>0, b>0$. Consider the following graph:



$y = ax_1 = bx_2 \Rightarrow x_2 = \frac{a}{b}x_1$
with $x_1 = \frac{y}{a}$, $x_2 = \frac{y}{b}$

Pick an arbitrary y_0 and consider the horizontal line at $x_2^0 = \frac{y_0}{b}$.
For $x_1 < \frac{y_0}{a}$ we have $y = ax_1$ so the MP_1 of input 1 is a.
For $x_1 > \frac{y_0}{a}$ we have $y = bx_2^0 = y^0$ so $MP_1 = 0$
There is a discontinuity at $x_1 = \frac{y_0}{a}$ so the MP_1 is not defined.
This is the cost-minimizing point for producing y^0.
The same sort of argument applies to MP_2 holding $x_1^0 = \frac{y_0}{a}$ fixed.

(b) The function is $y = x_1^\alpha x_2^\beta$ where $\alpha > 0$, $\beta > 0$.
We have IRS when $\alpha + \beta > 1$; CRS when $\alpha + \beta = 1$; and DRS when $\alpha + \beta < 1$. To show strict quasi-concavity, fix output at y_0 and look at the shape of the isoquant:



$$y^0 = x_1^{\alpha} x_2^{\beta} \Rightarrow x_2 = \left(\frac{y_0}{x_1^{\alpha}}\right)^{1/\beta}$$

$$x_2 = (y_0)^{1/\beta} x_1^{-\frac{\alpha}{\beta}}$$

$$\frac{dx_2}{dx_1} = (y_0)^{1/\beta}\left(-\frac{\alpha}{\beta}\right) x_1^{-\frac{\alpha}{\beta}-1} < 0$$

so the isoquant slopes down.

$$\frac{d^2 x_2}{dx_1^2} = (y_0)^{1/\beta}\left(-\frac{\alpha}{\beta}\right)\left(-\frac{\alpha}{\beta}-1\right) x_1^{-\frac{\alpha}{\beta}-2} > 0$$ so the isoquant flattens out as $x_1 \uparrow$ and $x_2 \downarrow$. This shows that the input requirement set $V(y)$ is strictly ~~quasi-concave~~ convex so the function is strictly quasi-concave (returns to scale are irrelevant).

(c) The function is $y = [x_1^{\rho} + x_2^{\rho}]^{1/\rho}$. Assume the firm minimizes cost so the FOC are

$$w_1 = \lambda \left(\tfrac{1}{\rho}\right)[x_1^{\rho} + x_2^{\rho}]^{1/\rho - 1} (\rho) x_1^{\rho-1}$$

$$w_2 = \lambda \left(\tfrac{1}{\rho}\right)[x_1^{\rho} + x_2^{\rho}]^{1/\rho - 1} (\rho) x_2^{\rho-1}$$

Divide the first equation by the second to get

$$\frac{w_1}{w_2} = \left(\frac{x_1}{x_2}\right)^{\rho-1} \Rightarrow \frac{x_1}{x_2} = \left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}}$$

Define the elasticity of substitution as $\sigma = -\dfrac{\partial\left(\frac{x_1}{x_2}\right)}{\partial\left(\frac{w_1}{w_2}\right)} \cdot \dfrac{\left(\frac{w_1}{w_2}\right)}{\left(\frac{x_1}{x_2}\right)}$

So

$$\sigma = -\left(\frac{1}{\rho-1}\right)\left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}-1} \cdot \frac{\left(\frac{w_1}{w_2}\right)}{\left(\frac{w_1}{w_2}\right)^{\frac{1}{\rho-1}}} = \frac{1}{1-\rho}$$

which is a constant (it does not depend on prices or input quantities, or the level of output).

2 (a) It is easy to find cases where profit maximization problems don't have solutions – for example, increasing returns to scale or constant returns when output price is above long run average cost. The problem in these cases is that the production possibilities set is unbounded, so profit is unbounded. For utility maximization, the set of feasible consumption bundles is bounded as long as prices and income are strictly positive. The feasible set is also normally closed, so it is compact. The utility function is continuous. We can therefore use the Weierstrasse Theorem to show that a maximum exists.

(b) A conditional input demand function is obtained by minimizing expenditure on inputs subject to a given output level. A Hicksian demand function is obtained by minimizing expenditure on goods subject to a given utility level. Mathematically, these are the same thing, so the functions have all the same properties.
This is not true for unconditional input demands and Marshallian demands. The unconditional input demands are obtained by maximizing profit subject to a technology constraint (with no budget constraint). Marshallian demands are obtained by maximizing utility subject to a budget constraint. This leads to different properties (for example there is an income effect for consumers, but not for firms).

2(c) The elasticity is $e(x) = \dfrac{\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i}{f(x)}$

This is important in the theory of the firm because it tells us whether returns to scale are locally increasing, constant, or decreasing and this tells us whether the long run average cost curve is locally falling, constant, or rising.
There is no similar concept in consumer theory because utility is an ordinal concept (all that matters is the ranking of consumption bundles) so we don't care about returns to scale. In general if we had a utility function with "DRS" we could find some increasing transformation that converts it into a new utility function with "IRS", or vice versa, without changing the rankings of the bundles.

3(a) Let x^* be the cost minimizing consumption bundle at prices p^* and utility u^*. Define the function
$g(p) = e(p, u^*) - px^* \leq 0$
where this is non-positive because x^* is not necessarily the cheapest way to get u^* at prices p.
However $g(p^*) = 0$ because x^* is the cheapest way to get u^* when the prices are p^*.
Because $g(p)$ is maximized at p^*, it satisfies the FOC for a max:

$$\frac{\partial g(p^*)}{\partial p_i} = \frac{\partial e(p^*, u^*)}{\partial p_i} - x_i^* = 0 \text{ for } i = 1 \ldots n.$$

By definition the Hicksian demand at (p^*, u^*) is x^*. Since (p^*, u^*) was arbitrary, we can remove the stars, which gives

$$\frac{\partial e(p,u)}{\partial p_i} = h_i(p,u) \text{ for all } i.$$

This is Shepherd's Lemma in the case of consumer theory.

(b) Start with the identity $u^* \equiv v[p, e(p,u^*)]$ which holds for all p.

Differentiate both sides with respect to p_i to get

$$0 = \frac{\partial v[p, e(p,u^*)]}{\partial p_i} + \frac{\partial v[p, e(p,u^*)]}{\partial m}\frac{\partial e(p,u^*)}{\partial p_i}$$

From Shepherd's Lemma, $\frac{\partial e(p,u^*)}{\partial p_i} = h_i(p,u^*)$ so

$$h_i(p,u^*) = -\frac{\frac{\partial v[p, e(p,u^*)]}{\partial p_i}}{\frac{\partial v[p, e(p,u^*)]}{\partial m}}$$

Let x^* maximize utility at (p^*, m^*) and let the resulting utility be u^*. Then

$$x_i^* = x_i(p^*, m^*) = h_i(p^*, u^*) = -\frac{\frac{\partial v(p^*, m^*)}{\partial p_i}}{\frac{\partial v(p^*, m^*)}{\partial m}}$$

where we substituted $m^* = e(p^*, u^*)$.

Removing the stars gives

$$x_i(p,m) = -\frac{\frac{\partial v(p,m)}{\partial p_i}}{\frac{\partial v(p,m)}{\partial m}}$$

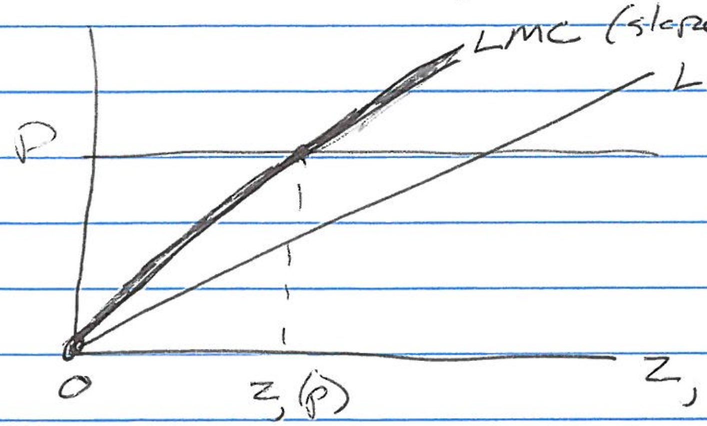
for all $i = 1 \ldots n$ which is Roy's Identity.

3(c). Use the Slutsky equation to study how the Marshallian demand $x_i(p,m)$ responds to p_i.

$$\frac{\partial x_i(p,m)}{\partial p_i} = \frac{\partial h_i[p, v(p,m)]}{\partial p_i} - \frac{\partial x_i(p,m)}{\partial m} \cdot x_i(p,m)$$

We know that the substitution effect $\frac{\partial h_i}{\partial p_i}$ is non-positive (This follows from the concavity of the expenditure function and Shephard's Lemma). So the only way to get $\frac{\partial x_i}{\partial p_i} > 0$ is by having $-\frac{\partial x_i}{\partial m} \cdot x_i(p,m) > 0$. Since $x_i(p,m) > 0$, we need $\frac{\partial x_i}{\partial m} < 0$, so good i must be inferior. Not only that but it must be so inferior that the income effect outweighs the substitution effect (ie it is a Giffen good). This is theoretically possible but not empirically likely.

4(a) $LAC = \frac{c(z_j)}{z_j} = z_j$ and $LMC = c'(z_j) = 2z_j$.



FOC: $p = 2z_j \Rightarrow z_j = \frac{p}{2}$

SOC holds because LMC is rising.

$S(p) = \frac{mp}{2}$ because there are m identical firms.

Profit will be positive for each firm because $p = LMC > LAC \Rightarrow$ profit $= (p - LAC) \times$ output > 0.

4(b) A typical consumer i solves

$$\max\ y_i + \ln x_i \quad \text{subject to} \quad px_i + y_i = w$$

or simply $\max \{w - px_i + \ln x_i\}$

FOC: $\frac{1}{x_i} - p = 0 \Rightarrow x_i = \frac{1}{p}$

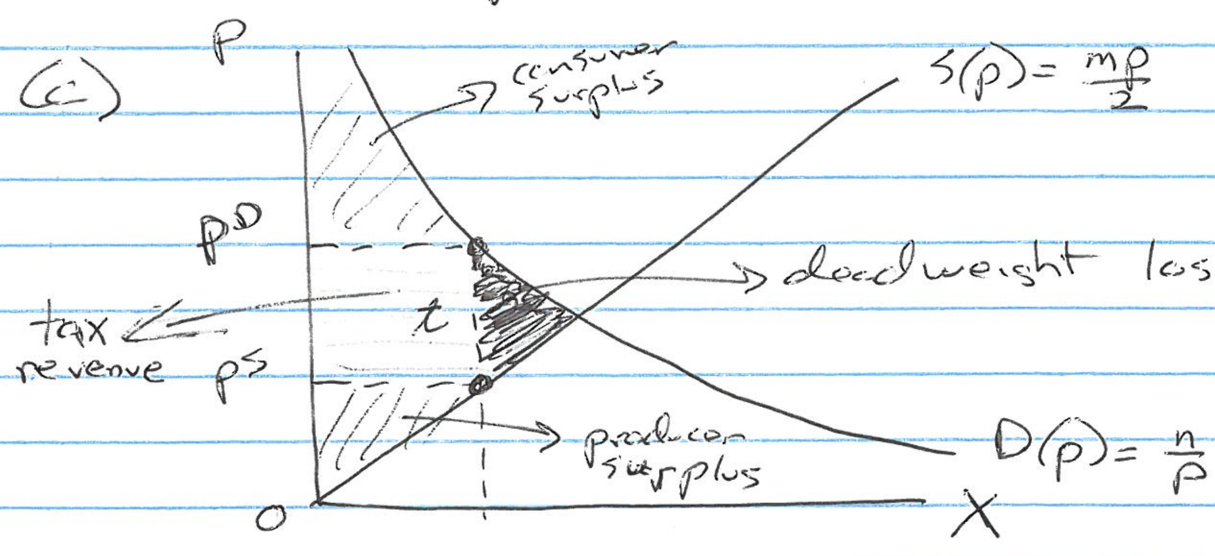
There are n identical consumers so $D(p) = \frac{n}{p}$

From part (a) we have $S(p) = \frac{mp}{2}$. Equating supply and demand $\Rightarrow \frac{mp}{2} = \frac{n}{p} \Rightarrow p^* = \sqrt{\frac{2n}{m}}$

Then $X^* = D(p^*) = \sqrt{\frac{nm}{2}}$

Yes, it is Pareto efficient. This is a model with quasi-linear utility and we showed in class that for such a model, the equilibrium allocation maximizes total utility (call the resulting total U^*). It is not possible to get a Pareto improvement because this would require making someone better off while not making anyone worse off. In this case total utility would have to exceed U^*, which is impossible because U^* is the max.

(c)



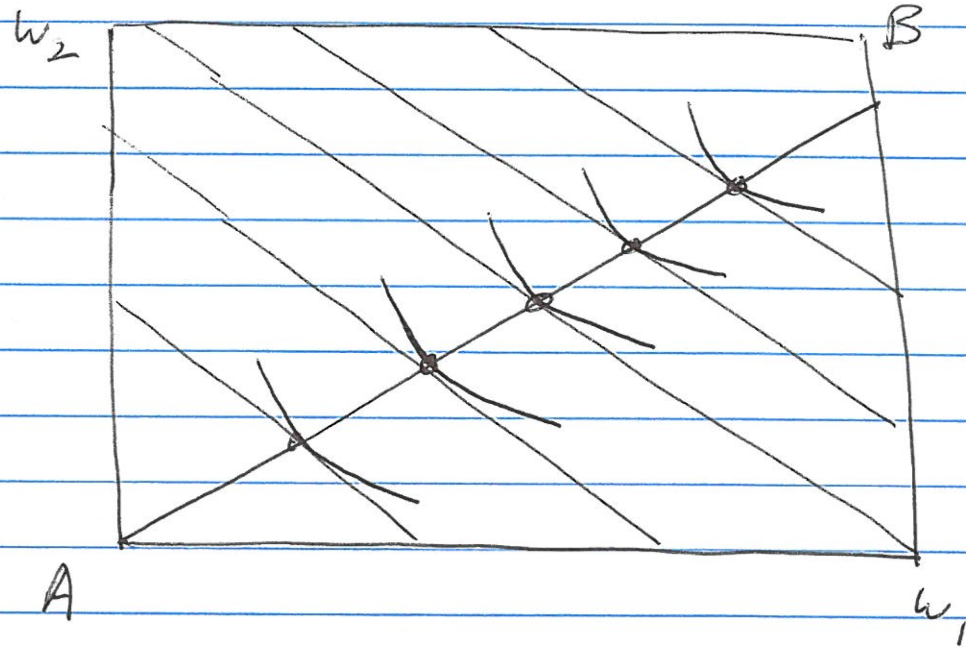
In the new equilibrium we have a demand price p^D and a supply price p^S where $p^D = p^S + t$. In equilibrium these prices have to give the same quantity of the X good. Substituting, we need

$$S(p^S) = \frac{mp^S}{2} = D(p^D) = \frac{n}{p^D} = \frac{n}{p^S + t}$$

This gives $\frac{mp^S}{2} = \frac{n}{p^S + t}$ or $p^S(p^S + t) = \frac{2n}{m}$

This is a quadratic in p^S, so we could solve it for p^S. Then compute $p^D = p^S + t$ to get the demand price.

5(a)



B's indifference curves are linear with slope $= -\frac{c}{d}$

A's indifference curves are strictly convex (they come from a Cobb-Douglas utility function). Along the contract curve the slopes must be equal: $MRS_A = MRS_B \Rightarrow$

$\boxed{\frac{x_{A2}}{x_{A1}} = \frac{c}{d}}$ so the contract curve is a ray from A's origin point with slope $= \frac{c}{d}$.

The contract curve does pass through origin A, but it does not necessarily pass through origin B. This would only occur if the point (w_1, w_2) happens to be on CC; that is, $\frac{w_2}{w_1} = \frac{c}{d}$.

(b) We know from the First Theorem that a WE is Pareto efficient, so the allocation must be on the contract curve. It is also obvious from the graph that the budget line will need to have the same slope as B's indifference lines (otherwise B would go to a corner solution that would not be on the contract curve). So we need to have

$$\boxed{\frac{p_1}{p_2} = \frac{c}{d}}$$

The equation for the contract curve gives $x_{A2} = \frac{c x_{A1}}{d} = \frac{p_1}{p_2} x_{A1}$

$\Rightarrow p_2 x_{A2} = p_1 x_{A1}$. A's budget constraint is

$$p_1 x_{A1} + p_2 x_{A2} = p_1 w_1$$

Combining these two equations gives

$$2p_1 x_{A1} = p_1 w_1 \Rightarrow \boxed{x_{A1} = \frac{w_1}{2} \text{ and } x_{A2} = \frac{w_1 c}{2d}}$$

$$2p_2 x_{A2} = p_1 w_1 \Rightarrow x_{A2} = \frac{p_1 w_1}{2p_2}$$

From market clearing, $\boxed{x_{B1} = \frac{w_1}{2} \text{ and } x_{B2} = w_2 - \frac{w_1 c}{2d}}$

Zero degree homogeneity is relevant because there are two prices, but we can only solve for the ratio $\frac{p_1}{p_2}$. Walras's Law is relevant because if we have prices such that $x_{A1} + x_{B1} = w_1$, then we automatically get $x_{A2} + x_{B2} = w_2$ and vice versa.

5(c) The planner maximizes $n_A\left[\ln x_{A1} + \ln x_{A2}\right] + n_B\left[c\,x_{B1} + d\,x_{B2}\right]$

Subject to

$$n_A x_{A1} + n_B x_{B1} = w_1$$
and
$$n_A x_{A2} + n_B x_{B2} = w_2$$

$$\Rightarrow \quad n_B x_{B1} = w_1 - n_A x_{A1}, \qquad n_B x_{B2} = w_2 - n_A x_{A2}$$

Substituting into the objective function, we want

$$\max \quad n_A \ln x_{A1} + n_A \ln x_{A2} + c\left[w_1 - n_A x_{A1}\right] + d\left[w_2 - n_A x_{A2}\right]$$

FOC:

$$\frac{n_A}{x_{A1}} - c\,n_A = 0 \quad \Rightarrow \quad \boxed{x_{A1} = \frac{1}{c}}$$

$$\frac{n_A}{x_{A2}} - d\,n_A = 0 \quad \Rightarrow \quad \boxed{x_{A2} = \frac{1}{d}}$$

$$\Rightarrow \quad \boxed{x_{B1} = \frac{w_1 - \frac{n_A}{c}}{n_B}} \quad \text{and} \quad \boxed{x_{B2} = \frac{w_2 - \frac{n_A}{d}}{n_B}}$$

To get a Walrasian equilibrium, as before we set $\frac{p_1}{p_2} = \frac{c}{d}$. Let the individual endowments be the same as the consumption bundles we want to achieve. Given that the budget line for each B consumer has the same slope as B's indifference curves, no B consumer wants to buy any other bundle – all of those consumers are maximizing utility by keeping their endowments. For the A consumers, check whether a typical individual is maximizing utility. The Marshallian demands for a person of type A are obtained by solving the following problem:

$\max \quad \ln x_{A1} + \ln x_{A2}$ subject to $p_1 x_{A1} + p_2 x_{A2} = m_A$
where m_A is income for a person of type A.
The usual methods give Marshallian demands

$$x_{A1}(p, m_A) = \frac{m_A}{2p_1} \qquad x_{A2}(p, m_A) = \frac{m_A}{2p_2}$$

Because the endowment vector for A is $\left(\frac{1}{c}, \frac{1}{d}\right)$ and the price vector can be $(p_1 = c,\ p_2 = d)$ – note that this is consistent with $p_1/p_2 = c/d$; we have
$m_A = p_1 w_{A1} + p_2 w_{A2} = c\left(\frac{1}{c}\right) + d\left(\frac{1}{d}\right) = 2.$

This implies in equilibrium $x_{A1}(p, m_A) = \frac{1}{p_1} = \frac{1}{c}$
$x_{A2}(p, m_A) = \frac{1}{p_2} = \frac{1}{d}$
This is identical to the endowment vector so it is utility maximizing for each person of type A to keep their endowment vector.

Finally, we need to check that both markets clear:
ie $n_A x_{A1} + n_B x_{B1} = w^1$ and $n_A x_{A2} + n_B x_{B2} = w^2$.
But both of these equations are satisfied due to the physical constraints on the social planner's problem.

Because ① all consumers are maximizing utility at the given prices and
② all markets clear,
we have constructed a Walrasian equilibrium that achieves the social planner's desired allocation.